

**RELATIVISTIC PARAMETRIZATION
OF THE P_{33}^1 RESONANCE IN PION-NUCLEON SCATTERING**

By

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**A Dissertation Presented to the Graduate Council
of the University of Florida
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A relativistic generalization of the Breit-Wigner formula is further improved and applied to the study of the P_{33}^1 resonance in pion-nucleon scattering. Reasonable fits with the P_{33} phase shift data are obtained. The new parametrization reduces to a simple Breit-Wigner form in the neighborhood of a resonance. Away from the resonance, however, the new parametrization has been seen to be far superior to the Breit-Wigner formula. This parametrization opens up a possible way of attacking the problem of hadron dynamics.

In addition, values of the coupling constants $C_{\pi N^1\Delta}$ and $C_{\pi\Delta\Delta}$ are predicted by this parametrization from phase shift fits.

CHAPTER I

INTRODUCTION

Direct channel resonances have been assumed to dominate the scattering amplitudes of strong interactions over a wide energy range. Empirically, this idea seems to be well supported.¹ Theoretically, it is now believed that the duality between direct channel resonances and cross channel exchanges allows the resonance-dominance assumption to coexist with the usual exchange mechanisms.²

In particular, the dominance of a nearby pole, plus the requirements of unitarity, have been used to describe the behavior of the scattering amplitude in the neighborhood of that pole. The most commonly used parametrization for a pole is the Breit-Wigner formula. This treatment, while exact at the pole, has the following deficiencies:³

- (1) It is non-relativistic.
- (2) Only open channels are used, nearby singularities can thus be omitted.
- (3) Poles are produced on all sheets of the scattering amplitude.
- (4) The position of the pole is unrelated to partial widths, thus the parametrization does not say anything concerning the dynamical origin of the resonance.

Previous works by Parkinson and others have succeeded in building up a relativistic K-matrix formalism which is very similar to the Breit-Wigner formula in form, but does not have any of the above objections.^{3,4}

This work represents an extension of the new parametrization, which has been rather successful in describing the rho meson, to the case of a high-spin baryon resonance, the P_{33}^1 in pion-nucleon scattering. It is intended to be the first in a series of parametrizations of direct channel poles to be used as a rigorous check of duality. It is also hoped that the series of works can shed some light on the significance of the pomeron.

In order to make a systematic study of various resonances, the phase space function has been studied in detail. It is found that if the widths of resonances are ignored, i.e., if there is no "spread" in the masses of resonances, it is possible to write down simple general phase space functions for all channels involving such particles. In this work the zero width assumption has been used for certain channel thresholds. The refinement of non-zero widths will be attempted later.

Using recent phase shifts of Carter et al.⁵ and a compilation of older results,⁶ the parametrization has been used to determine the mass and width of the P_{33}^1 resonance, as well as the coupling constants $C_{\pi N\Delta}$, $C_{\pi\Delta\Delta}$ and $C_{\pi N^*\Delta}$.

General aspects of pion-nucleon scattering and its kinematics will be studied in Chapter II. In Chapter III the

general parametrization formula will be developed. Chapter IV deals principally with the phase space function which plays an important role in the parametrization. Chapter V studies the problem of parametrization of the P'_{33} . It also includes a discussion of data handling. The last chapter sums up the findings of this work.

The notations of reference 4 will be followed. The P'_{33} is sometimes denoted by Δ for short.

CHAPTER II

PION-NUCLEON SCATTERING

1. Introduction

The interaction between pions and nucleons is one of the fundamental problems of subatomic physics.

For nuclear physics, the determination of the nuclear potential, the long-range component of which is dominated by the interaction between pions and nucleons, is to be a giant step forward. It will bring theoretical nuclear physics to the same level of development as present day atomic and molecular physics, where the many-body problem is the only outstanding obstacle.

To particle physics, pion-nucleon scattering is especially important because of the availability of high intensity pion beams at a wide energy range. The large amount of data collected from pion-nucleon scattering not only serves as a check on most theoretical concepts in hadron physics, it is also the source of many new ideas.

In particular, phase shift analyses of partial wave amplitudes for this scattering process have produced evidence for a series of new particles or resonances, and this discovery has been extended to other scattering processes. The

presence of a large number of resonances, all of which are considered as essential constituents as the pion and the nucleon, is a new feature only found in particle physics. Their presence provides very severe constraints on all dynamical models of strong interactions.

Various detailed dynamical models have found different degrees of success in describing experimental results. Notable among low energy models are the Chew-Low theory and the effective range approximation. Among high energy models are Regge pole theory and the Veneziano model. The low energy theories will be briefly discussed later in this chapter.

2. Kinematics

In the present work only Lorentz-invariant dynamical variables will be used. The relationship between the invariants and ordinary physical variables such as energy and momentum is established here.

In addition, only two-body reactions will be considered.

From a practical point of view, this restriction is unavoidable because three- or more-body kinematics is complicated and the kinematic singularities of corresponding helicity amplitudes are not known.

A theoretical implication of this restriction will be taken up later.

A typical reaction is, therefore,

$$1 + 2 \longrightarrow 3 + 4$$

where particles $i=1,\dots,4$ have four-momenta p_i , rest masses m_i , spins s_i , and helicity components λ_i . If particles 1 and 2 are a pion and a nucleon, corresponding to the initial state of the scattering, the above reaction will be called the direct channel or s channel. By contrast, the cross channels are

$$1 + \bar{3} \longrightarrow \bar{2} + 4 \quad (t \text{ channel})$$

$$1 + \bar{4} \longrightarrow \bar{2} + 3. \quad (u \text{ channel})$$

The terminologies will be made reasonable in a moment.

In the direct channel, particles 3 and 4 may be any one of a number of combinations. They may be the pion and the nucleon in the case of elastic scattering, or the pion and the Δ , the pion and the P_{11}' resonance, in the case of inelastic scattering. These different combinations will also be called "channels", or more correctly, the exit channels.

The basic Lorentz-invariant dynamical variable in the direct channel is the total center-of-mass energy squared,

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \quad (1)$$

where the p_i are measured in the direct channel. The momentum transfers t and u are defined by

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \quad (2)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \quad (3)$$

The three invariants are not independent,

$$s + t + u = \sum_i m_i^2 \quad (4)$$

where the summation is over the incident and the exit channels.

If θ is the scattering angle in the center-of-mass frame of the direct channel, then

$$\sin \theta = 2[s\phi(s,t)]^{1/2}/S_{12}(s)S_{34}(s) \quad (5)$$

$$\cos \theta = [s^2 + 2st - s \sum_i m_i^2 + (T_{12}U_{12}T_{34}U_{34})^{1/2}]/S_{12}(s)S_{34}(s) \quad (6)$$

where

$$\begin{aligned} \phi(s,t) = & stu - s(T_{13}U_{13}T_{24}U_{24})^{1/2} - t(T_{12}U_{12}T_{34}U_{34})^{1/2} + \\ & - (m_1^2 + m_4^2 - m_2^2 - m_3^2)(m_1^2 m_4^2 - m_2^2 m_3^2) \end{aligned} \quad (7)$$

$$T_{ij} = (m_i + m_j)^2 \quad (8)$$

$$U_{ij} = (m_i - m_j)^2 \quad (9)$$

$$S_{ij}(s) = [(s - T_{ij})(s - U_{ij})]^{\frac{1}{2}}. \quad (10)$$

In the two-dimensional space spanned by s , t and u , not all points are "physical". In other words, there are regions in this space that are inaccessible for realistic scattering processes. This follows from the fact that the four-momenta are constrained by

$$p_i^2 = m_i^2. \quad (11)$$

The boundary of the physical region is given by

$$\phi(s, t) = 0. \quad (12)$$

All square roots in the above are chosen to be positive in the physical region, so that the sine of θ is positive in the direct channel.

T_{ij} is to be called the threshold of the exit channel with particles i and j , and U_{ij} is the corresponding pseudo-threshold.

In the center-of-mass frame of the direct channel, the total energy is

$$W = s^{\frac{1}{2}} \quad (13)$$

and the magnitude of the three-momentum of either particle i or particle j is

$$k_{ij} = S_{ij}(s)/2W. \quad (14)$$

In the laboratory system, where the nucleon target is at rest, the total energy of the incident pion is

$$E = (s - m^2 - \mu^2)/2m \quad (15)$$

where m is the nucleon mass and μ the pion mass.

3. The Scattering Amplitude

An experiment in strong interactions consists of observing the initial state of two or more particles, allowing them to interact, and then observing the final states of the arbitrary number of particles resulting. The S-matrix element, $\langle f | \underline{S} | i \rangle$, is defined such that

$$P_{fi} = \langle i | \underline{S}^\dagger | f \rangle \langle f | \underline{S} | i \rangle \quad (16)$$

represents the probability of $|f\rangle$ being the final state, given the initial state $|i\rangle$.

It is well established that strong interactions have very short ranges. Since all weak but long-range interactions such as electromagnetism and gravitation are ignored

in theoretical investigation and subtracted out of experimental results, the finite range of strong interactions guarantees that $|f\rangle$ and $|i\rangle$ are free particle states under ordinary experimental situations. In the following all particles in the incident and exit channels are regarded as free, and suitable kinematics is applied.

The helicity components of particles in a given channel can be used instead of the spin projection along a fixed axis in space for a complete description of hadron states. When this is done, a helicity amplitude can be defined in terms of the S-matrix. In particular, the helicity amplitude $h_{\lambda\mu}$ is defined by

$$S_{\lambda\mu} = \langle p_3, p_4, \lambda_3, \lambda_4 | p_1, p_2, \lambda_1, \lambda_2 \rangle + \frac{i}{2\pi} h_{\lambda\mu} \quad (17)$$

where

$$\lambda = \lambda_3 - \lambda_4$$

$$\mu = \lambda_1 - \lambda_2$$

$$S_{\lambda\mu} = \langle p_3, p_4, \lambda_3, \lambda_4 | \underline{S} | p_1, p_2, \lambda_1, \lambda_2 \rangle.$$

The differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\lambda\mu} = |h_{\lambda\mu}|^2 / k^2 \quad (18)$$

where k is the magnitude of the center-of-mass three momentum in the direct channel, for the incident particles.

The helicity amplitude $h_{\lambda\mu}$ defined here is related to the helicity amplitude of Jacob and Wick⁷ by

$$h_{\lambda\mu} = k f_{\lambda\mu}^{JW} \quad (19)$$

and to the usual helicity amplitude such as seen in Wang⁸ and King⁹ by

$$h_{\lambda\mu} = \frac{1}{2\pi} \left[\frac{kq}{s} \right]^{\frac{1}{2}} f_{\lambda\mu} \quad (20)$$

where q is the magnitude of the center-of-mass three-momentum in the exit channel.

Partial wave expansion for $S_{\lambda\mu}$ and $h_{\lambda\mu}$ are defined as follows:

$$S_{\lambda\mu} = \frac{1}{4\pi} \sum_J (2J + 1) S_{\lambda\mu}^J d_{\lambda\mu}^J(\theta) e^{-i(\lambda-\mu)\phi} \quad (21)$$

$$h_{\lambda\mu} = \sum_J (2J + 1) h_{\lambda\mu}^J d_{\lambda\mu}^J(\theta) e^{-i(\lambda-\mu)\phi} \quad (22)$$

where $d_{\lambda\mu}^J(\theta)$ is the d function of the rotation matrix element

$$d_{\lambda\mu}^J(\theta) = \langle J\lambda | e^{-iR\theta} | J\mu \rangle. \quad (23)$$

The total cross section, according to equations (18) and (22), is therefore given by

$$\sigma_{\lambda\mu} = \frac{4\pi}{k^2} \sum_J (2J + 1) |h_{\lambda\mu}^J|^2. \quad (24)$$

4. The Phase Shift

According to equation (16), and since there must be unit probability for an initial state to end up in some final state,

$$\begin{aligned} 1 &= \sum_f \langle i | \underline{S}^\dagger | f \rangle \langle f | \underline{S} | i \rangle \\ &= \langle i | \underline{S}^\dagger \underline{S} | i \rangle. \end{aligned} \quad (25)$$

Here the completeness of final states $|f\rangle$ has been assumed. If, in addition, as is customary, the initial state is normalized, then it follows that

$$\underline{S}^\dagger \underline{S} = \underline{1} \quad (26)$$

since the initial state $|i\rangle$ is arbitrary. This is the unitarity relation for the scattering matrix.

Combining equations (21) and (26), it is found that the unitarity relation for the partial wave is

$$\underline{S}^{J\dagger} \underline{S}^J = \underline{1}. \quad (27)$$

In the case of elastic pion-nucleon scattering, the

partial wave scattering matrix has only four helicity components:

$$S_{++}^J \quad S_{+-}^J \quad S_{-+}^J \quad S_{--}^J$$

where $+$ denotes $+\frac{1}{2}$ and $-$ denotes $-\frac{1}{2}$. By parity invariance, it is found that the first equals the last, and the second equals the third. Using these relations, the unitarity relation for elastic scattering becomes

$$\left| S_{++}^J \right|^2 + \left| S_{+-}^J \right|^2 = 1 \quad (28)$$

$$S_{++}^{J*} S_{+-}^J + S_{+-}^{J*} S_{++}^J = 0.$$

Define

$$S_-^J = S_{++}^J + S_{+-}^J \quad (29)$$

$$S_+^J = S_{++}^J - S_{+-}^J$$

and the unitarity relation becomes even simpler:

$$\left| S_-^J \right|^2 = \left| S_+^J \right|^2 = 1. \quad (30)$$

Conventionally, one insures that this condition is satisfied by writing

$$S_{\pm}^J = e^{2i\delta_{J\pm}} \quad (31)$$

where the phase shift $\delta_{J\pm}$ is a real function of W .

In terms of the helicity amplitudes,

$$\begin{aligned}
 \frac{h_{++}^J}{4\pi k} &= \frac{S_{++}^J - 1}{2ik} \\
 &= \frac{1}{4ik} [e^{2i\delta_{J-}} + e^{2i\delta_{J+}} - 2] \\
 &= \frac{1}{2} [h_-^J + h_+^J]
 \end{aligned} \tag{32}$$

where

$$h_{\pm}^J(W) = \frac{1}{k} e^{i\delta_{J\pm}(W)} \sin \delta_{J\pm}(W). \tag{33}$$

A similar expression can be obtained for h_{+-}^J .

Above the first inelastic threshold, the phase shift can no longer be real. The unitarity relation now reads

$$\langle i | \underline{S}^\dagger | i \rangle \langle i | \underline{S} | i \rangle = 1 - \sum_{n \neq i} \langle i | \underline{S}^\dagger | n \rangle \langle n | \underline{S} | i \rangle. \tag{34}$$

The phase shift must therefore have a positive imaginary part. However, it is conventional to factor out the imaginary part, in the form of an absorption parameter η , and make the phase shift real even for inelastic processes.

Hence equation (31) becomes

$$S_{\pm}^J = \eta_{J\pm} e^{2i\delta_{J\pm}}. \tag{35}$$

$\eta_{J\pm}$ is equal to unity when the scattering is purely elastic, and it is smaller than unity when the scattering is partially inelastic.

5. The Breit-Wigner Formula

From the definition of h_{\pm}^J , equation (33), it can be found that, if inelasticity can be ignored,

$$h_{\pm}^J = \frac{1}{k} \frac{1}{\cot \delta_{J\pm} - i}. \quad (36)$$

A resonance is said to exist at W_r if $\cot \delta_{J\pm}$ goes through zero for this value of W with a negative derivative. If a power series expansion of $\cot \delta_{J\pm}$ is made, then in the neighborhood of W_r

$$\cot \delta_{J\pm} = (W_r - W)/\frac{1}{2}\Gamma \quad \Gamma > 0 \quad (37)$$

and

$$h_{\pm}^J = \frac{1}{k} \frac{\Gamma/2}{W_r - W - i\Gamma/2}. \quad (38)$$

Inserting this result into equation (24) and neglecting the other partial waves, the resonance cross section is found to be

$$\sigma_{\text{res}}(W) = \frac{4\pi}{k^2} (2J + 1) \frac{\Gamma^2/4}{(W_r - W)^2 + \Gamma^2/4}. \quad (39)$$

This is the well-known Breit-Wigner formula.

At resonance energy, the partial-wave cross section is,

according to the Breit-Wigner formula,

$$\sigma_{\text{res}}(W_r) = \frac{4\pi}{k^2} (2J + 1) \quad (40)$$

which is commonly called the unitarity limit, since this is the absolute maximum of a partial-wave cross section by equation (33).

It should be pointed out that in the derivation of equation (39), the fact that h_{\pm}^J and h_{++}^J (h_{+-}^J) have different normalization is used.

6. The Chew-Low Model

The Chew-Low model is the first model that successfully predicted the gross characteristics of pion-nucleon scattering at pion energies below a few hundred MeV.¹⁰ In particular, it predicted the P_{33}^+ resonance.

A non-relativistic pseudovector interaction between pions and nucleons is assumed in this model, from which, with the additional assumption of a nucleon at rest, the interaction hamiltonian of the following form is obtained:

$$H_I = \sum_m (V_m a_m + V_m^\dagger a_m^\dagger) \quad (41)$$

$$V_I = i f (\vec{\sigma} \cdot \vec{k} / [2E]^{1/2}) \tau_m n(k^2) \quad (42)$$

where a_m^\dagger and a_m are respectively the creation and annihilation operators for single pions, \vec{k} is the three momentum and

E the energy of the pion, and m represents all pion quantum numbers, and lastly, $n(k^2)$ is the Fourier transform of the nucleon charge density. In order to ensure the convergence of the necessary integrations $n(k^2)$ must be cut off above some k_{\max} .

By virtue of the pseudovector interaction and the assumption that the nucleon remains rigidly at rest, the model predicts that all scatterings will be in the P-state.

The Low scattering equation is solved with the assumption that the only two diagrams of importance are

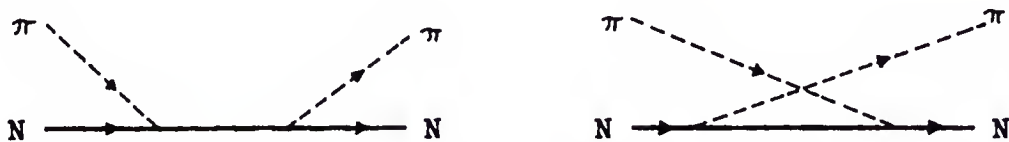


Figure 1. Diagrams

Finally, the following relation is derived for the $I=\frac{3}{2}$ $J=\frac{3}{2}$ phase shift

$$\frac{k^3}{E} \cot \delta_{P33} = \frac{3}{4f^2} (1 - rE) \quad (43)$$

where f is the renormalized coupling constant and r a constant approximately given by

$$r \cong f^2 E_{\max} > 0. \quad (44)$$

E_{\max} is the pion energy corresponding to k_{\max} .

Relations similar to equation (43) are predicted for the other P-wave phase shifts. However, for these phase

shifts $r < 0$. This means that there are no resonances in these partial waves. Experimentally the other phase shifts are small in the low energy region.

The Chew-Low model was a break-through for field theory. It was the first calculation involving a strong interaction which gave any significant agreement with experiment. By its specific assumption it is limited to low energies. It is also a particular case of the effective range approximation, which will be discussed next.

7. The Effective Range Approximation

It is convenient for the discussion of effective range approximation to use the partial wave scattering amplitudes $h_{\pm}^J(w)$ defined in equation (33). Here the new variable w , the square of k , is preferred over W the total energy. From the definition, it is found that

$$\text{Im } h_{\pm}^J(w) = k |h_{\pm}^J(w)|^2 \quad (45)$$

if inelasticity can be ignored. From equations (36) and (45) it follows that

$$\text{Im} \left[\frac{1}{h_{\pm}^J(w)} \right] = -k. \quad (46)$$

It is possible to seek a power series expansion in the low energy region of w by using analyticity. However, the expansion cannot be made simply for $h_{\pm}^J(w)$, for they have a

branch point at $w=0$. The function $M(w)$ is defined as

$$M_{\pm}^J(w) = 1/h_{\pm}^J(w) + ik. \quad (47)$$

$M(w)$ is analytic in the neighborhood of $w=0$, since $\text{Im } M = 0$ for w real and positive, and therefore by the Schwarz reflection principle $M(w)$ has no discontinuity across the positive real axis. Furthermore, in the physical region,

$$M_{\pm}^J(w) = k \cot \delta_{J\pm}. \quad (48)$$

Thus the power series expansion will take the following form, where the parameters are conventional:

$$k \cot \delta_J = -\frac{1}{a} + \frac{1}{2}rw + \dots \quad (49)$$

a is commonly called the scattering length and r the effective range. The effective range approximation is made by keeping only the first two terms in the expansion.

Because of threshold behavior of helicity amplitudes, about which a later chapter will be devoted to, equation (49) is not always the most convenient expansion to make. A general form commonly seen is

$$k^{2J+1} \cot \delta_J = -\frac{1}{a} + \frac{1}{2}rw. \quad (50)$$

CHAPTER III

THE K-MATRIX FORMALISM

In a parametrization of the dynamics of scattering processes it is most desirable to take care of all kinematic effects explicitly in the formalism. In a dispersion theory these effects are generated by kinematic singularities of the scattering amplitudes.

It will be assumed that these kinematic singularities factorize, i.e., for the scattering from channel i to channel f , the helicity amplitude can be written as

$$h^J(f,i) = [\rho_i^J(s)]^{\frac{1}{2}} M^J(f,i) [\rho_f^J(s)]^{\frac{1}{2}} \quad (51)$$

where $\rho_x^J(s)$ depends only on dynamical variables and characteristics of channel x and $M^J(f,i)$ contains only dynamical singularities. The function $\rho_x^J(s)$ will be called the J -th partial-wave phase space function for channel x , or simply the phase space function for channel x if J is understood. The phase space function is real above the channel threshold and it vanishes below the threshold.

By their definitions in equations (21) and (22), it is found that

$$\underline{S}^J = \underline{1} + 2i\underline{h}^J. \quad (52)$$

Combining equations (27) and (52), the following relation is obtained:

$$\text{Im} \left[\frac{1}{\underline{h}^J} \right] = -\underline{1}. \quad (53)$$

If it is defined that

$$[\rho_{mn}^J]^{\frac{1}{2}} = [\rho_m^J(s)]^{\frac{1}{2}} \delta_{mn} \theta(s - T_m) \quad (54)$$

where T_m is the threshold in channel m and $\theta(s-T_m)$ is the step function, then

$$\underline{M}^J = [\underline{\rho}^J]^{-\frac{1}{2}} \underline{h}^J [\underline{\rho}^J]^{-\frac{1}{2}} \quad (55)$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{J,fi} = \frac{1}{k^2} \rho_f^J \rho_i^J |M^J(f,i)|^2 \quad (56)$$

$$\text{Im} \left[\frac{1}{\underline{M}^J} \right] = -\underline{\rho}^J. \quad (57)$$

The last equation is the key to the K-matrix formalism. If, furthermore, we define

$$R_{ij}^J(s) = \left[\frac{1}{\pi} \int_{T_i}^{\infty} \frac{\rho_i^J(s') ds'}{s' - s - i\epsilon} + \text{p.s.} \right] \delta_{ij} \quad (58)$$

where p.s. stands for "possible subtraction", then it follows from equation (57) that a K-matrix can be defined as follows:

$$\frac{1}{\underline{\underline{K}}^J} = \frac{1}{\underline{\underline{M}}^J} + \underline{\underline{R}}^J \quad (59)$$

and that it is real for positive real s . In addition, if both $\underline{\underline{\rho}}^J$ and $\underline{\underline{M}}^J$ are analytic in s , so is $\underline{\underline{K}}^J$. Equation (39) can also be written as

$$\underline{\underline{M}}^J = \frac{\underline{\underline{K}}^J}{1 - \underline{\underline{K}}^J \underline{\underline{R}}^J}. \quad (60)$$

Equation (53) has been obtained as a general result of unitarity. It can be obtained more readily from equation (46) and the relation between the two helicity amplitudes. However, the result obtained this way is valid only for the low energy region.

By definition $\underline{\underline{R}}^J$ is a diagonal matrix. The diagonal elements of this matrix will sometimes be simply denoted by $R_i^J(s)$.

At this point the following physical considerations can be made:

- (1) For the sake of resonance study, it is only necessary to use a single incident channel. Mathematically, this means that both $\underline{\underline{M}}$ and $\underline{\underline{K}}$ are column matrices. (Note the helicity indices have been suppressed in favor of channel indices.)
- (2) Near a resonance, elements of the K-matrix are assumed to have the simple form

$$\underline{\underline{K}} = \frac{\underline{\underline{C}}}{s - s_r} \quad (61)$$

where s_r is the position of the resonance, and $\underline{\underline{C}}$ is a matrix

of coupling constants that factorizes:

$$C_{ij} = C_i C_j. \quad (62)$$

Equation (61) is the "pole approximation" of the scattering amplitude and it contains the only dynamical singularity assumed for the formalism. The factorization property of the residue of the pole corresponds to the non-degeneracy of the resonance.

(3) In the neighborhood of a resonance, the M-matrix elements have a form similar to the expression (61). That is, the amplitude should indicate a simple pole at the resonance energy. To ensure that this is the case, according to equation (59), \underline{R}^J should be at least quadratic in $(s-s_r)$ in the neighborhood of s_r .

Hence, equation (58) should be changed to be

$$R_{ij}^J(s) = \left[\frac{1}{\pi} (s-s_r)^2 \int_{T_i}^{\infty} \frac{\rho_i^J(s') ds'}{(s'-s_r)^2 (s'-s-i\epsilon)} + \text{p.s.} \right] \delta_{ij}. \quad (63)$$

Combining the above considerations, the following parametrization formula is found:

$$\underline{M}^J(s) = \frac{\underline{C}}{(s_r-s) - \sum_j C_j^2 R_j^J(s) - i \sum_j C_j^2 \rho_j^J(s) \theta(s-T_j)}. \quad (64)$$

CHAPTER IV

PHASE SPACE FUNCTION

In order to have a well-defined parametrization formula, it remains to find a unique expression for the phase space function. The phase space function for a two-body channel is a product of two factors: the regular two-body phase space function for spinless particles, ρ_0 , as required by kinematics, and the kinematic singularities of the helicity amplitude, which is caused by the presence of spins.

1. The Regular Two-Body Phase Space Function

The regular phase space function for spinless particles is well-known.¹¹ It is the Jacobian of the transformation from the momentum space to the phase space or invariant space. In a two-body channel, the Jacobian depends only on a single variable, s .

By definition, in the center-of-mass frame,

$$d\rho_0 = \frac{2}{\pi} d^4p_1 d^4p_2 \delta^4(p_1 + p_2 - P) \delta(p_1^2 - m_1^2) \delta(p_2^2 - m_2^2) \quad (65)$$

$$\text{or} \quad \rho_0 = \frac{1}{s} [(s - T_{12})(s - U_{12})]^{\frac{1}{2}} \quad (66)$$

where T_{12} and U_{12} are respectively the threshold and the pseudothreshold of the channel consisting of particles 1 and 2. The normalization of this function and the normalization of the coupling constants are closely related. In order to avoid confusion, ρ_0 has been defined with no numerical coefficient here. The coupling constant is going to carry an overall normalization.

The presence of spins does not change the Jacobian from the momentum space to the invariant space. But it introduces kinematic singularities in the helicity amplitudes.

2. Kinematic Singularities of Helicity Amplitudes

The kinematic singularity structure of helicity amplitudes has been studied extensively.^{8,9,12,13} It includes singularities at (a) the boundary of the physical region, (b) the threshold and the pseudothreshold, and (c) the origin of the s -plane.

(a) The Boundary of the Physical Region

The d function of the rotation matrix is related to the Jacobi polynomials by

$$d_{\lambda\mu}^J(\theta) = \pm \left[\frac{(J+M)! (J-M)!}{(J+N)! (J-N)!} \right]^{\frac{1}{2}} [\cos \frac{1}{2}\theta]^{|\lambda+\mu|} [\sin \frac{1}{2}\theta]^{|\lambda-\mu|} \cdot P_{(J-M)}^{(|\lambda-\mu|, |\lambda+\mu|)}(\cos \theta) \quad (67)$$

where $M = \max(|\lambda|, |\mu|)$ and $N = \min(|\lambda|, |\mu|)$. From the definition of the partial-wave expansion it can be seen that $h_{\lambda\mu}$ contains the factors

$$[\cos \frac{1}{2}\theta]^{|\lambda+\mu|} [\sin \frac{1}{2}\theta]^{|\lambda-\mu|}. \quad (68)$$

By equations (5) and (12), the above expression represents a series of zeros or singularities at the boundary of the physical region. These are called t zeros or singularities, because for given s , the positions of these zeros or singularities are determined by the scattering angle through t , the momentum transfer in the direct channel.

Since only direct channel kinematic singularities are of interest in the parametrization, these possible singularities at the boundary of the physical region, or cross channel singularities, are ignored in this work.

(b) The Threshold and the Pseudothreshold

The covariant helicity operator approach of King and others is employed here.⁹

The direct channel covariant helicity operators corresponding to particles 1 and 2 are defined by the relations

$$F_1^S = 2 w_\mu^{(1)} p_2^\mu / S_{12}(s) \quad (69)$$

$$F_2^S = 2 w_\mu^{(2)} p_1^\mu / S_{12}(s) \quad (70)$$

where μ is a summation index and $W_\mu^{(i)} W^{(i)\mu}$ is the Casimir operator for the spin of particle i . In the s channel center-of-mass frame these operators reduce to the ordinary helicity operators. Similarly, the t channel covariant helicity operators for particles 1 and 2 are defined by

$$F_1^t = -2 W_\mu^{(1)} p_3^\mu / S_{13}(t) \quad (71)$$

$$F_2^t = 2 W_\mu^{(2)} p_4^\mu / S_{24}(t). \quad (72)$$

They also reduce to helicity operators in the t channel center-of-mass frame.

The transformations between the eigenstates of the two sets of operators are given by

$$\psi^S(p_r, \lambda) = \sum_{\lambda'} d_{\lambda' \lambda}^{J_r}(\chi_r) \psi^t(p_r, \lambda') \quad (73)$$

where λ and λ' are eigenvalues of the operators F_r^S and F_r^t , respectively, J_r is the intrinsic spin of particle r , and $r=1$ or 2 . χ_2 is the angle between $-\vec{p}_1$ and $-\vec{p}_4$ in the rest frame of particle 2, while χ_1 is the angle between $-\vec{p}_2$ and \vec{p}_3 in the rest frame of particle 1. The coordinate axes are always chosen such that $0 \leq \chi_r \leq \pi$. In terms of the invariants s and t , the angles χ_r are given by the relations

$$\cos \chi_1 = \frac{(s + m_1^2 - m_2^2)(t + m_1^2 - m_3^2) - 2m_1^2 M}{S_{12}(s) S_{13}(s)} \quad (74)$$

$$\cos \chi_2 = - \frac{(s + m_2^2 - m_1^2)(t + m_2^2 - m_4^2) + 2m_2^2 M}{S_{12}(s) S_{24}(t)} \quad (75)$$

where $M = m_4^2 - m_2^2 + m_1^2 - m_3^2$

and

$$\sin \chi_1 = \frac{2m_1 [\phi(s, t)]^{\frac{1}{2}}}{S_{12}(s) S_{13}(t)} \quad (76)$$

$$\sin \chi_2 = \frac{2m_2 [\phi(s, t)]^{\frac{1}{2}}}{S_{12}(s) S_{24}(t)}. \quad (77)$$

From their definitions the operators F_1^s and F_2^s can be seen to be ill-defined at the threshold and the pseudo-threshold. The covariant helicity amplitudes, $f_{\lambda\mu}^s$, being eigenstates of these operators, are singular at these values of s . Introducing a new set of amplitudes $f_{\lambda\mu}^{st}(s, t)$ in which the states for particles 3 and 4 remain unchanged but the states for 1 and 2 are now eigenstates of F_1^t and F_2^t , it is found that

$$\begin{aligned} f_{\lambda\mu}^s(s, t) &= f_{\lambda_3 \lambda_4 \lambda_1 \lambda_2}^s(s, t) \\ &= \sum_{\lambda' \mu'} d_{\lambda' \lambda_1}^{J_1}(\chi_1) d_{\mu' \lambda_2}^{J_2}(\chi_2) f_{\lambda_3 \lambda_4 \lambda' \mu'}^{st}(s, t). \end{aligned} \quad (78)$$

The singularities at the threshold and the pseudothreshold are now isolated in the d functions, because $f_{\lambda_3 \lambda_4 \lambda' \mu'}^{st}(s, t)$ is regular at these points.

After an analysis of the asymptotic behavior of the d functions, it is found that the amplitudes $f_{\lambda\mu}^s$ behave as⁹

$$[s - T_{34}]^{-\frac{1}{2}(J_3+J_4)} \quad (79)$$

and

$$[s - U_{34}]^{-\frac{1}{2}(J_3+J_4)} \quad (80)$$

near the final-state threshold and pseudothreshold respectively. Thus the product of the expressions in the above is the singularity to be included in the phase space function. However, it should be noted here that the kinematic singularity of the helicity amplitude is, by equation (51), the square root of $\rho_f^J(s)$, but not $\rho_f^J(s)$ itself.

(c) The Origin

In the above discussion the origin has not been found to be a singular point of the helicity amplitude, but it is often pointed out that the origin is a singular point of the scattering amplitude. Cohen-Tannoudji et al.¹² concluded that the helicity amplitude does not have singularities at this point. It is only the linear combinations of helicity amplitudes commonly used for Reggeization that are singular here. This is also pointed out by King, since the singularity of a linear combination can be different from the singularity of a single helicity amplitude.⁹

3. The Phase Space Function

The product of the expressions in equations (66), (79)

and (80) is not quite the phase space function yet. The phase space function has been defined for partial wave amplitudes only. Referring to equation (22), it is seen that the singularity of $d_{\lambda\mu}^J(\theta)$ has to be studied, too. According to the asymptotic behavior for the d functions,^{9,14}

$$\begin{aligned} d_{\lambda\mu}^J(\theta) &\sim [\cos \theta]^J \\ &\sim [(s - T_{34})(s - U_{34})]^{-\frac{1}{2}J} \end{aligned} \quad (81)$$

because of equation (6). Thus, finally,

$$\rho^J(s) = \frac{1}{s} [(s - T_{34})(s - U_{34})]^{J+\frac{1}{2}-J_3-J_4}. \quad (82)$$

It must be pointed out that the threshold and pseudo-threshold singularities have been obtained for particles with well-defined masses, and the parametrization is dealing with channels that often involve many unstable particles or resonances. In using equation (82) it is assumed that the widths of the contributing channels are sufficiently small to be ignored. This is a limitation of the present formalism.

CHAPTER V

PARAMETRIZATION OF THE P_{33}^1 RESONANCE

1. Introduction

The parametrization formula gives general representations of all matrix elements of a given partial-wave helicity amplitude. The theory does not have the power to predict the exact behavior of individual matrix elements. The matrix elements can only be determined by "parametrization" against experimental data.

The P_{33} phase shifts, here chosen as the only experimental data input, can be expected to determine one particular matrix element. However, because of the way phase shifts are defined in equation (33), it turns out that a linear combination of two matrix elements is determined instead. The P_{33} phase shift is related to $h_{-}^{\frac{1}{2}}$, which is proportional to the linear combination $h_{++}^{\frac{1}{2}} + h_{+-}^{\frac{1}{2}}$. This raises two questions.

First, since singularities of certain linear combinations of helicity amplitudes are known, is it not a better way to start directly with the parametrization of the proper linear combination? Theoretically, this is an attractive alternative. In fact, it is not known a priori that a parametrization developed for $h_{\lambda\mu}^J$ can produce an adequate ap-

proximation for $h_{\lambda\mu}^J + h_{\lambda,-\mu}^J$, because they may have different kinematic singularities. But practically, direct parametrization of linear combinations is very unattractive because the kinematic singularities involved are such that a higher number of subtractions is usually needed, in which case the calculation may be obscured by the presence of a large number of subtraction constants. Other linear combinations simply do not have known singularities, so that it is impossible to use such an approach even if the subtraction constants are not an objection. In pion-nucleon scattering, both difficulties are present.

Second, since the linear combination $h_{\lambda\mu}^J - h_{\lambda,-\mu}^J$, or $h_{+}^{\frac{1}{2}}$, is related to the D_{33} phase shift, is it not advisable to make a simultaneous parametrization of the P_{33}' and D_{33} resonances, so that individual matrix elements, i.e., $h_{\lambda\mu}^J$ and $h_{\lambda,-\mu}^J$, might be determined? The answer is definitely yes, it would seem. But the helicity amplitudes themselves are not physically important for the strong interaction, because they are not parity-invariant. The physically important amplitudes are the parity-invariant linear combinations.

2. Basis of Computation

In Table I are listed the various channels that contribute to the P_{33}' resonance, their characteristics, and the appropriate phase space functions.

The phase space functions for the $J=\frac{3}{2}$ channels are ob-

Table I. Characteristics of Contributing Channels

Meson	Baryon	$T^{\frac{1}{2}}$ [MeV]	$U^{\frac{1}{2}}$ [MeV]	ρ
pion	nucleon	1077.9	843.7	$\frac{1}{s} [(s-T)(s-U)]^{3/2}$
pion	P_{33}^1	1373.6*	1094.4*	$\frac{1}{s} [(s-T)(s-U)]^{\frac{1}{2}}$
pion	P_{11}^1	1609.6	1340.4	$\frac{1}{s} [(s-T)(s-U)]^{3/2}$

* varied in actual computation as the P_{33}^1 mass is changed.

tained from equation (82). While for the pion- P_{33}^1 channel only two subtractions are required to make the dispersion integral convergent, in the other channels three are needed. Thus the dispersion integral for the second channel has the form of equation (63), without the p.s. term, the same integrals for the other channels are of the following form

$$R_j^J(s) = \frac{1}{\pi} (s - s_r)^3 \int_{T_i}^{\infty} \frac{\rho_j^J(s') ds'}{(s' - s_r)^3 (s' - s - i\epsilon)}. \quad (83)$$

The real part of the denominator in equation (64) is denoted by $D(s)$, and the imaginary part by $G(s)$. They are:

$$D(s) = s_r - s + z(s - s_r)^2 - \sum_j c_j^2 R_j^J(s) \quad (84)$$

$$G(s) = - \sum_j c_j^2 \rho(j) \theta(s - T_j) \quad (85)$$

where z is an overall subtraction constant that is made necessary by the extra subtraction, and $\theta(s)$ is a step function.

The phase shift is given by

$$\tan \delta_J(s) = - \frac{G(s)}{D(s)}. \quad (86)$$

Because of the fact that most coupling constants involved in the parametrization of the P_{33}^1 are unknown at the moment, one of the objectives of this work is to make certain predictions on their values. The procedure is to use phase shift data and the parametrization formula to find a best fit, with the coupling constants and the subtraction constant, as well as the mass of the resonance, as free parameters. The computer routine PARFIT at the Department of Physics and Astronomy of the University of Florida is used for this purpose.

3. Phase Shifts

Two sets of phase shifts have been used. Set A is taken from Carter et al.⁵ Set B is compiled from older results.^{6,15}

The Cavendish-Rutherford collaboration of Carter et al. recently reported an extensive series of cross section measurements at the CERN synchrocyclotron in the energy range around the P_{33}^1 resonance. The reported error bars are much smaller than all previously available results. An interesting feature of the new data is an apparent shift of the peak of the cross section to a lower mass. The phase shifts re-

ported by this group are extracted directly from the total cross sections.

Set A is used to yield more accurate information on the P_{33}^1 resonance: its mass and width, and possibly also its coupling constants.

In Set B, values of phase shift reported by Moorhouse, Kirsopp, Johnson, Donnachie, Bareyre et al.¹⁵ and a set from Berkeley called Path 1⁶ are taken on equal footings and their average determined. Since there is no available basis to prefer any one set of these over any other, no weight factors have been assigned in the average.

Since different listings cover different energy ranges, the number of entries at any particular energy can vary considerably. In the lower range where entries typically number two or three, the standard deviation is determined by the following observation:

In the Saclay isovector pion-pion phase shifts,¹⁶ the quoted uncertainty roughly forms a band of constant width about the mean when plotted graphically.

In the absence of further information, this observation has been applied to assign standard deviations for set B. In the higher energy range where data entries are more numerous, the standard deviation is taken to be one half of the maximum difference among the sets, generally. And in order to achieve some kind of uniformity throughout the energy range, the width of the band in the lower range has been determined at the upper energy range.

The results are listed in column 2 of Table IV.

4. Left Half-Plane Singularities

In the complex s -plane, the partial-wave helicity amplitude has dynamical singularities at poles and branch cuts corresponding to all "communicating" channels or possible exit channels as required by unitarity.¹⁴ These are known as the right-hand singularities for they occur in the right half-plane. There are also left-hand singularities, occurring in the left half-plane, because of cross channel right-hand singularities.

The N/D method of Ball and Parkinson, which is equivalent to a K-matrix formalism using the R-matrix of equation (63), which in turn approximates the contribution of right-hand singularities, uses one form of the effective range approximation for the left-hand singularities.³ With an increasing number of subtractions in the R-matrix, it has been found that more left-hand poles are generated. In particular, the R-matrix of equation (83) generates a pair of complex conjugate left-hand poles. The symmetric poles about the left-hand branch cuts along the negative real axis is a better approximation than a single pole. A wider range of validity can be expected than the effective range approximation.

This line of reasoning has been used in works on the approximation of a Regge cut by a pair of complex conjugate

Regge poles in the J-plane.¹⁷

In this respect, it should be noted that if the parametrization is to extend to higher and higher energies, resonances of higher and higher J values will have to be included. Then, by equation (82), the phase space function will be so modified that more and more subtractions are necessary in the R-matrix. And this indeed is going to extend the region of validity of the parametrization to higher and higher energies.

Returning to the present problem, it is seen that the energy range is relatively small. Thus a two-pole approximation should be sufficient.

5. Contributing Channels

In equation (64) the summations are over all contributing channels. These are also called communicating channels or possible exit channels. In principle they cannot be exhausted, unless experimentalists find an upper limit for resonance production. Practically, the choice is made in the following manner.

From the set of well-established particles, i.e., the ones listed by the Particle Data Group with complete quantum number and mass-width specifications,¹⁸ all meson-baryon pairs that have the right quantum number combinations are selected, and their thresholds and pseudothresholds, according to equations (8) and (9), calculated. Most thresholds

lie beyond the top of the energy range of phase shift data.

Referring to equation (64), it is found that channels whose thresholds are higher than the top of the energy range under consideration contribute only to the function $D(s)$, through the R-matrix. Elements of the R-matrix are relatively small for those channels with high thresholds, as can be seen from equation (83). Here a high threshold means a large $s'-s$ in the denominator of the integrand, and consequently a small matrix element. Thus all channels with very high thresholds are not considered important, and a total of eight channels are finally selected as possible contributing channels. These are listed in Appendix B as a comment in the computer program actually used in the calculation.

The maximum energy for the phase shift data is at 1672 MeV. Arranged in order of increasing thresholds, the fourth channel threshold is at $(1660 \text{ MeV})^2$ and the eighth at $(1783 \text{ MeV})^2$. In fitting the phase shift data it is found that all channels higher than the third can be ignored without affecting the chi-square of the fits. Thus, for the energy range under consideration, there are only three contributing channels. These are listed in Table I.

The practical aspect of two-body kinematics has been studied in Chapter II. From a theoretical point of view, the exclusive use of two-body channels is an application of the idea of direct channel resonance dominance. That is, three- or more-body channels, though experimentally observed to be dominating at times, are themselves dominated by two-

body channels.

As an example, in pion-nucleon scattering the pion production process of two pions and a nucleon in the exit channel are often observed. Rather than trying to solve a three-body problem, the view is taken that the "extra" pion has come mostly from either a rho meson or any one of the baryon resonances. (The word "come" must be understood in a very loose sense.) Thus, the inclusion of pion-baryon resonance and rho meson-nucleon channels should describe the three-body channel adequately.

Direct channel dominance, used in this manner, is quite similar to Feynman diagrams in field theory, though there is a subtle difference. In Feynman diagrams, the scattering is assumed to have gone through a virtual intermediate state. In the S-matrix theory there is merely a statement of dominance in the amplitude, and a subsequent substitution.

6. Computation

The general procedure of actual computation is as follows:

A set of values for the P_{33}^1 resonance mass and the coupling constants in various channels, plus the subtraction constant, is supplied to the program PARFIT, which uses equation (86) to determine the chi-square value on the basis of experimental phase shifts and standard deviations. PARFIT also automatically minimizes the chi-square by adjusting the

free parameters, that is, the coupling constants and the subtraction constant. At the end of computation PARFIT produces the best fit, with all relevant data.

The mass of the P_{33}^1 is then shifted and a new best fit found. The best overall fit with the mass as a parameter is taken to be the final result.

In addition, the number of contributing channels can be varied in each chi-square calculation, by varying the value of PAR.

Thus, the K-matrix formalism provides the most general form for the parametrization. The number of contributing channels is determined by the sensitivity to chi-square fit, and the actual parameters determined by best fit against experimental data. And a complete parametrization for the P_{33}^1 is obtained.

The mechanics of actual calculation are contained in the Appendices. The routine PARFIT is not included.

CHAPTER VI

RESULTS AND CONCLUSIONS

1. Comparison with Breit-Wigner Formula

The K-matrix can be made relativistic simply by using phase space functions and dynamical variables that are all Lorentz invariant. This has been done in Chapters II, III, and IV. Thus the parametrization is relativistic. And this removes the first objection.

It should be remarked here that it is not the aesthetic aspect of non-relativistic nature of the Breit-Wigner formula that prompted the objection. Rather, it is the fact that non-relativistic mechanics restricts all formalisms to low energies, or narrow energy ranges, and this restriction is quite undesirable.

The summation in the parametrization formula, equation (64), in the real part of the denominator specifically, includes both open and closed channels for any particular energy. Hence nearby singularities can influence the dynamics of the scattering.

The zeros of the denominator in equation (64) do not appear on all sheets of the scattering amplitude, in general.¹⁹

The zero of the denominator in equation (64) is determined, in part, by the expression

$$\sum_j C_j^2 R_j^J(s). \quad (87)$$

The relative contribution of each channel toward producing the resonance is then given by³

$$C_j^2 R_j^J(s_r). \quad (88)$$

This gives a rough indication of the relative importance of each channel in the dynamical origin of the resonance.

2. General Description of Phase Shift Fits

Set A

Reasonable fits to phase shift data have been obtained. The best chi-square value for 14 points is 15.3, corresponding to a resonance mass of 1230.4 MeV. Results of the parametrization fit are listed in Table II. Table III contains the phase shifts and the values of chi-square at all data points, together with the experimental data used.

It is of interest to note that the lowest data point carries the worst chi-square. If this data point is deleted, the best chi-square value would be 10.7 for 13 points. Previously, the data point at the other end of the spectrum has been considered a bad fluctuation.¹⁸

Table II. Parameters from Phase Shift Fits

	Set A	Set B
resonance mass	1230.4 MeV	1235.0 MeV
resonance width [*]	120 MeV	122 MeV
$C_{\pi N \Delta}$	0.1551 ± 0.0002	0.1543 ± 0.0010
$C_{\pi \Delta \Delta}$	0.0830 ± 0.0073	0.1029 ± 0.0073
$C_{\pi N' \Delta}$	0.8048 ± 0.0202	1.018 ± 0.031
subtraction constant	-0.0488 ± 0.0003	-0.0506 ± 0.0016
chi-square	15.31	63.68
mean chi-square	1.09	1.63

^{*} full width at half maximum.⁴

With a total of five de facto parameters (three coupling constants, one subtraction constant, and the resonance mass), the fit to set A has not been very good.¹⁸ But it should be pointed out that different channels contribute differently to the resonance, and not all coupling constants affect the quality of the fit equally significantly. In fact, the introduction of some more channels and their coupling constants does not change the chi-square value at all. The parametrization used here is not an optimum mathematical approximation.

Set B

The best chi-square value for the 39-point set B is

Table III. Phase Shifts Fitted According to Carter et al.⁵

W [MeV]	Exp. [deg.]	Theory	χ^2
1139	11.87±0.15	11.54	4.75
1160	21.41±0.06	21.42	0.01
1177	33.17±0.10	33.36	3.50
1178	34.36±0.20	34.26	0.25
1190	45.64±0.11	45.66	0.04
1193	47.83±0.11	47.81	0.04
1206	62.82±0.18	62.61	1.31
1210	67.73±0.21	67.48	1.37
1215	73.51±0.30	73.63	0.15
1227	84.94±0.98	85.97	1.10
1244	102.05±0.51	102.3	0.21
1261	114.41±0.23	114.6	0.45
1280	124.03±0.17	124.2	0.85
1301	131.96±0.15	131.8	1.27

63.7. This result is as good as can be expected, because systematic errors among different sources are expected to be important. For this very reason results of this fit should not be taken too seriously, especially those pertaining to the P'_{33} resonance. In Table II are listed main findings of this fit. In Table IV are the phase shifts.

Similar to the parametrization of the rho meson,⁴ in this relatively wide energy range the theoretical values deviate from experimental phase shifts significantly near the

Table IV. Phase Shifts Fitted According to Compilation Data

W [MeV]	Exp. [deg.]	Theory	χ^2
1094	1.4 ± 0.9	0.88	0.33
1104	2.6 ± 0.9	2.01	0.42
1109	3.6 ± 0.9	2.76	0.88
1113	4.2 ± 0.9	3.44	0.71
1127	7.6 ± 0.9	6.66	1.32
1160	21.0 ± 1.0	19.79	1.47
1177	31.8 ± 1.1	31.12	0.39
1185	37.9 ± 1.1	37.77	0.00
1195	45.9 ± 1.3	47.28	1.12
1197	48.2 ± 1.4	49.32	0.64
1202	54.2 ± 1.4	54.59	0.08
1213	65.8 ± 1.4	66.71	0.52
1216	69.4 ± 1.4	70.05	0.21
1231	85.1 ± 1.4	86.09	0.61
1235	90.1 ± 1.4	90.04	0.00
1247	100.6 ± 1.4	100.7	0.04
1252	105.2 ± 1.3	104.7	0.15
1254	107.5 ± 1.2	106.2	1.20
1258	109.4 ± 1.0	109.0	0.07
1268	117.7 ± 1.0	115.4	5.16
1275	119.8 ± 1.0	119.3	0.23
1291	126.9 ± 0.6	126.8	0.02
1320	136.7 ± 1.3	136.8	0.01
1362	144.9 ± 0.6	146.5	7.33
1390	149.2 ± 1.0	151.2	3.86
1416	153.3 ± 1.1	154.6	1.81
1442	156.3 ± 1.1	157.5	1.28
1470	160.2 ± 1.4	160.2	0.00
1481	161.1 ± 1.7	161.1	0.00
1500	164.5 ± 2.5	162.6	0.59
1512	166.0 ± 0.9	163.4	8.14
1524	166.1 ± 1.0	164.2	3.45

Table IV. continued

W [MeV]	Exp. [deg.]	Theory	χ^2
1543	168.3 + 2.2	165.4	1.69
1572	170.3 + 2.8	167.1	1.31
1601	172.2 + 3.0	168.6	1.41
1617	171.2 + 1.7	169.3	1.30
1629	173.2 + 4.0	169.0	1.11
1658	174.1 + 3.6	167.6	3.24
1672	175.5 + 2.6	166.7	11.56

top of the range. The chi-square for a 38-point fit, for example, would be 52.1, a significant improvement. The introduction of higher contributing channels does not change this top-of-the-range deviation.

Since set B covers a wide energy range, in the calculation of the chi-square values the number of contributing channels has been varied. The result is that only the lowest three channels contribute, the same channels as present for phase shift data set A. Altogether eight channels, with thresholds up to $(1783 \text{ MeV})^2$, have been tried. It may be conjectured that a certain channel with still higher threshold contributes significantly by virtue of a huge coupling constant. At least this cannot be ruled out yet. And this very high threshold may solve the difficulty at the higher end of the range.

3. The Resonance Mass

Set A

The mass of 1230.4 MeV for the P_{33}^1 resonance is lower than most values reported.¹⁸ The lone exception is the "nuclear" result reported by Carter et al.⁵ themselves. The shift to lower values is in agreement with most recent works. However, it is worth noting that various forms of the Breit-Wigner formula, all based on the same experimental data, consistently give higher masses.¹⁸

The full width at half maximum is 120 MeV.

Set B

The resonance mass is high compared with the one for set A. This is typical of older results. It is believed that, due to higher systematic and statistical errors, information concerning the P_{33}^1 extracted from this fit is of very limited value.

4. The Subtraction Constant

The subtraction constant is very small, both for set A and for set B. It is believed that the magnitude of the subtraction constant is a measure of the goodness of fit, too. For example, in the successful rho meson parametrization, it has been found that not only is the chi-square low, the subtraction constant is practically zero. A physical

interpretation is the following: the subtraction contributes to offset any deficiency in the knowledge of kinematic and dynamic effects. Had the last two effects been properly handled, there would be no need for a non-zero subtraction constant. This is reinforced by a mathematical interpretation of the subtraction: it is merely a free term whose function is to reduce the chi-square.

5. The Coupling Constants

A. The Pion-Nucleon- P_{33}^1 Coupling

Each term in the expansion for $-G(s)$ can be identified with a certain partial decay width for the resonance, if it is evaluated at the resonance energy.³ In particular, using the value of $C_{\pi N \Delta}$ obtained in the parametrization, we have

$$\Gamma = C_{\pi N \Delta}^2 \frac{1}{m_{\Delta}^2} \left[\left\{ m_{\Delta}^2 - (m_N + m_{\pi})^2 \right\} \left\{ m_{\Delta}^2 - (m_N - m_{\pi})^2 \right\} \right]^{\frac{1}{2}}$$

$$= 0.81 \quad (89)$$

in pion mass unit. This can be compared with the full width at half maximum of 0.87. They represent two different ways in which the width can be defined.

According to Mathews,²¹ this partial decay width is also given by

$$\Gamma = \frac{g_{\pi N\Delta}^2}{4\pi} \frac{2}{3} p^3 \frac{(m_\Delta + m_N)^2 - m_\pi^2}{m_\Delta^2}. \quad (90)$$

Comparing the last two expressions, it can be seen that

$$C_{\pi N\Delta}^2 = \frac{g_{\pi N\Delta}^2}{4\pi} \frac{(m_\Delta + m_N)^2 - m_\pi^2}{48 m_\Delta^3}. \quad (91)$$

Or that a $C_{\pi N\Delta}$ of 0.1551 corresponds to a $\frac{g_{\pi N\Delta}^2}{4\pi}$ of 0.37, in dimensionless unit.

Ebel et al.²⁰ report a $\frac{g_{\pi N\Delta}^2}{4\pi}$ of 0.33, which they maintain is inferior to the value of 0.34 obtained from non-relativistic spinless Born approximation. By inference, it would seem that the value of 0.37 is quite acceptable.

B. The Pion- P_{33}^1 - P_{33}^1 Coupling

According to Rushbrooke,²² the partial width corresponding to the decay of a P_{33}^1 resonance into a pion and a P_{33}^1 resonance is given by

$$\begin{aligned} \Gamma &= \frac{g_{\pi\Delta\Delta}^2}{4\pi} \frac{p}{3m_\Delta} \left[3 \left(\frac{m_\Delta^2 + m_\Delta^2 - m_\pi^2}{2m_\Delta} - m_\Delta \right) + \frac{2p^2}{3m_\pi^2} \left(\frac{m_\Delta^2 + m_\Delta^2 - m_\pi^2}{2m_\Delta} - 2m_\Delta \right) \right] \\ &\sim \frac{g_{\pi\Delta\Delta}^2}{4\pi} \frac{p}{m_\Delta} \frac{m_\pi^2}{2m_\Delta} \end{aligned} \quad (92)$$

since the center-of-mass momentum of the decay product is very small. According to the K-matrix formalism, the same quantity is given by

$$\Gamma = C_{\pi\Delta\Delta}^2 \frac{2p}{m_{\Delta}}. \quad (93)$$

Combining the two we have

$$C_{\pi\Delta\Delta}^2 = \frac{g_{\pi\Delta\Delta}^2}{4\pi} \frac{m_{\pi}^2}{4m_{\Delta}}. \quad (94)$$

Our value of $C_{\pi\Delta\Delta} = 0.083$ therefore corresponds to the value 0.243 for $g_{\pi\Delta\Delta}^2/4\pi$ in dimensionless unit.

Hori and Kanki²³ reported that $g_{\pi\Delta\Delta}^2$ is approximately one ninth of $g_{\pi N\Delta}^2$. The relativistic SU(6) model of Sakita and Wali,²⁴ on the other hand, predicted that $g_{\pi\Delta\Delta}^2$ is about nine times of $g_{\pi N\Delta}^2$. Sutherland²⁵ and Michael²⁶ predicted that $g_{\pi\Delta\Delta}^2$ is even larger by a factor of about two. Our result is not in close agreement with any of the above. $g_{\pi\Delta\Delta}^2$ is shown here to be two-thirds of $g_{\pi N\Delta}^2$.

C. The Pion- P_{11}^{\prime} - P_{33}^{\prime} Coupling

The K-matrix formalism developed here cannot be applied to the decay of the P_{11}^{\prime} into a pion and a P_{33}^{\prime} resonance, though the appropriate branching ratio and total width have been measured experimentally. Thus no comparison can be made at this point.

It is worth noting that a similar problem occurs in reference 22. The decay of a spin- $\frac{1}{2}$ particle into a spin- $\frac{3}{2}$ particle and a spin-0 particle warrants special equivalence relations for satisfactory results.

6. General Remarks

In spite of its many known and suspected deficiencies, the formalism developed here not only points to a way by which some experimental results can be derived through a theoretical model, it also opens up an approach by which the whole problem of hadron dynamics may be attacked.

Traditionally, hadron physics has two distinct components, the low energy and the high energy theories. Duality is a way to reconcile the two components. The present approach, on the other hand, is to extend the low energy theory into higher and higher energies. If the K-matrix parametrization scheme is successful there, a unified theory of hadron physics is in sight. And it may eventually be able to check the validity of duality, and of many other interesting concepts popular in high energy physics.

Of more practical concern, and related to the quality of these phase shift fits, is the fact that phase shifts must be deduced from "direct" experimental data through some kind of parametrization first. Existing parametrizations commonly used are all based on the Breit-Wigner formula. In this light the chi-square values mentioned in this work cannot be too small. The present work offers an alternative to the variations of the Breit-Wigner formula. Because of the many objections of the latter as studied in Chapter I, and their removal in the K-matrix formalism, it is believed that many difficulties associated with present-day description of

the phase shift results can be removed.

APPENDIX A
MECHANICS OF CALCULATION

The basic principal integral is defined by³

$$f(s, T, U) = P \int_T^{\infty} [(s' - T)(s' - U)]^{-\frac{1}{2}} \frac{ds'}{s' - s} \quad (A1)$$

so that

$$\text{for } s > T, \quad f(s, T, U) = -2[(s - T)(s - U)]^{-\frac{1}{2}} \coth^{-1} \left[\left(\frac{s - U}{s - T} \right)^{\frac{1}{2}} \right] \quad (A2)$$

$$\text{for } T > s > U, \quad f(s, T, U) = 2[(T - s)(s - U)]^{-\frac{1}{2}} \tan^{-1} \left[\left(\frac{s - U}{T - s} \right)^{\frac{1}{2}} \right] \quad (A3)$$

$$\text{for } U > s, \quad f(s, T, U) = 2[(T - s)(U - s)]^{-\frac{1}{2}} \tanh^{-1} \left[\left(\frac{U - s}{T - s} \right)^{\frac{1}{2}} \right] \quad (A4)$$

and the derivative of $f(s, T, U)$ with respect to s is

$$f'(s, T, U) = [(s - T)(s - U)]^{-1} \left[-1 + \frac{1}{2}(T + U - 2s)f(s, T, U) \right] \quad (A5)$$

which may be used iteratively to produce higher derivatives with respect to the same variable.

APPENDIX B COMPUTER PROGRAM

C	channel #	meson	baryon	psf	T	U
C	1	pion 140	ncln 940	3	1080	800
C	2	pion 140	delta 1234	1	1374	1094
C	3	pion 140	N1470 1470	3	1610	1330
C	4	pion 140	N1520 1520	1	1660	1380
C	5	pion 140	N1535 1535	3	1675	1395
C	6	kaon 496	sigma 1190	3	1686	694
C	7	rho 765	ncln 940	1	1705	175
C	8	eta 549	delta 1234	1	1783	685

C 1. energy unit in pion mass for all calculations
 C 2. input energy unit is the MeV
 C 3. normalization of energy by variable "unit"
 C 4. A(par) is the subtraction constant
 C 5. numerical differentiation is used
 C 6. exp data input: UCRL20030 compilation

```

implicit real*8 (a-h,k-z)
real x
integer psf
common/data/unit,m,T(8),U(8),psf(8)/intg/w(39,8),energy(
+39)/stpdx/fx,x(400)/mass/pion,kaon,rho,eta,ncln,delta,
+N1470,N1520,N1535,sigma
read (1,1) eta,kaon,pion,rho,delta,N1470,N1520,N1535,
+ncln,sigma
1 format (10d8.1)
read (1,2) psf
2 format (8i7)
do 4 i=1,39

```

```

      read (1,3) energy(i)
3  format (2x,d8.1)
4  continue
      do 5 i=1,400
5  x(i)=0.1
      call callit
      stop
      end

      double precision function f(A,L)
      implicit real*8 (a-h,k-z)
      real rgm,A(1),dum1,dum2
      integer psf,L,par
      cnvr=180./3.14159265358979d0
      common/data/unit,m,T(8),U(8),psf(8)/intg/w(39,8),energy(
+39)/stpdf/fx/srchit/rgm(50),dum1(2554),par,dum2(9),inx
+/mass/pion,kaon,rho,eta,ncln,delta,N1470,N1520,N1535,
+sigma
      if (inx.ne.1) go to 15
      inx=2
      read (1,11) delta
11  format (d12.4)
      write (3,20) delta
      unit=pion
      T(1)=((ncln+pion)/unit)**2
      U(1)=((ncln-pion)/unit)**2
      T(2)=((delta+pion)/unit)**2
      U(2)=((delta-pion)/unit)**2
      T(3)=((N1470+pion)/unit)**2
      U(3)=((N1470-pion)/unit)**2
      T(4)=((N1520+pion)/unit)**2
      U(4)=((N1520-pion)/unit)**2
      T(5)=((N1535+pion)/unit)**2
      U(5)=((N1535-pion)/unit)**2
      T(6)=((sigma+kaon)/unit)**2
      U(6)=((sigma-kaon)/unit)**2

```

```

T(7)=((ncln+rho)/unit)**2
U(7)=((ncln-rho)/unit)**2
T(8)=((delta+eta)/unit)**2
U(8)=((delta-eta)/unit)**2
m=(delta/unit)**2
isum=par-1
do 14 i=1,39
s=(energy(i)/unit)**2
do 12 j=1,isum
12 w(i,j)=v(j,s)
write (3,13) (w(i,j), j=1,isum)
13 format (8g16.3)
14 continue
15 i=rgm(1)
s=(rgm(2)/unit)**2
g=0.d0
isum=par-1
do 16 j=1,isum
if (s.lt.T(j)) go to 16
g=g-A(j)**2*fcf(psf(j),s,T(j),U(j))
16 continue
d=m-s+A(par)*(m-s)**2
do 17 j=1,isum
17 d=d+A(j)**2*w(i,j)
h=dabs(d)
if (h.lt.1.d-20) go to 19
god=-g/d
if (god.lt.0) go to 18
f=datan(god)*cnvr
fx=f
return
18 f=180.0+datan(god)*cnvr
fx=f
return
19 f=90.d0
fx=f

```

```

    return
20  format ('1',g50.6)
    end

    double precision function fcn ( dmm1,dmm2,dmm3,dmm4)
    implicit real*8 (a-h,o-z)
    integer dmm1
    go to (21,22,23), dmm1
21  fcn=dsqrt(( dmm2-dmm3)*( dmm2-dmm4))/dmm2
    return
22  fcn=0.d0
    return
23  fcn=( dsqrt(( dmm2-dmm3)*( dmm2-dmm4)))**3/dmm2
    return
    end

    double precision function v(i,r)
    implicit real*8 (a-h,m-z)
    integer psf
    common/data/unit,m,T(8),U(8),psf(8)
    j=psf(i)
    p=T(i)
    z=U(i)
    a=m-p
    b=m-z
    e=m-r
    x=r-p
    y=r-z
    pi=3.14159265358979d0
    cutoff=9.9d25
    go to (41,42,43), j
41  g=a*b/m/e
    gp=g*( 1.d0/a+1.d0/b-1.d0/m-1.d0/e)
    v=(-p*z*e**2*q(0.d0,p,z,cutoff)/r/m**2+x*y*q(r,p,z,cutoff
+ )/r+e**2*gp*q(m,p,z,cutoff)+e**2*g*qp(m,p,z,cutoff))/pi
    return

```

```

42  v=0.d0
    return
43  h=a**2*b**2/m/e
    hp=h*(2.d0/a+2.d0/b-1.d0/m-1.d0/e)
    hpp=hp*(2.d0/a+2.d0/b-1.d0/m-1.d0/e)+h*(1.d0/m**2+1.d0/
+e**2-2.d0/a**2-2.d0/b**2)
    v=(p**2*z**2*e**3*q(0.d0,p,z,cutoff)/r/m**3-x**2*y**2*q(
+r,p,z,cutoff)/r+e**3*hpp*q(m,p,z,cutoff)/2.d0+e**3*hp*qp
+(m,p,z,cutoff)+e**3*h*qpp(m,p,z,cutoff)/2.d0)/pi
    return
end

```

```

double precision function q(s,t,u,lmda)
implicit real*8 (a-z)
factor=(lmda-t)/(lmda-u)
if (s.lt.u) go to 32
if (s.eq.u) go to 33
if (s.lt.t) go to 31
q=-1./dsqrt((s-t)*(s-u))*dlog((1.+dsqrt((s-t)/(s-u)/
+factor))/(1.-dsqrt((s-t)/(s-u)/factor)))
return
31  q=2./dsqrt((t-s)*(s-u))*datan(dsqrt((s-u)/(t-s)*factor))
    return
32  q=1./dsqrt((s-u)*(s-t))*dlog((1.+dsqrt((s-u)/(s-t)*
+factor))/(1.-dsqrt((s-u)/(s-t)*factor)))
    return
33  q=2./(t-u)*dsqrt(factor)
    return
end

```

```

double precision function qp(s,t,u,lmda)
implicit real*8 (a-z)
if (s.eq.u) go to 34
factor=dsqrt((lmda-u)*(lmda-t))/(s-lmda)
qp=(factor+((t+u)/2.-s)*q(s,t,u,lmda))/((s-u)*(s-t))
return

```

```

34  factor=dsqrt((lmda-t)/(lmda-u))
    qp=2./3.*factor/(t-u)*(2./(t-u)+1./(lmda-u))
    return
    end

double precision function qpp(s,t,u,lmda)
implicit real*8 (a-z)
if (s.eq.u) go to 35
qpp=((t+u)/2.-s)*qp(s,t,u,lmda)-q(s,t,u,lmda)*(1.+((t+u
+)/2.-s)*(1./(s-t)+1./(s-u)))-dsqrt((lmda-t)*(lmda-u))/(s
+-lmda)*(1./(s-t)+1./(s-u)+1./(s-lmda)))/((s-t)*(s-u))
    return
35  factor=dsqrt((lmda-t)/(lmda-u))
    qpp=4./15.*factor/(t-u)**2*(8./(t-u)+4./(lmda-u)-3.*(t-u
+)/((lmda-u)**2))
    return
    end

```

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BIOGRAPHICAL SKETCH

Yin-po Tschang, in Chinese 章印波, was born in Chungking, China on June 20, 1944. He received his early education in Taipei, Singapore, and Hong Kong, before entering Chung Chi College of the Chinese University of Hong Kong on a four-year Hong Kong Government Scholarship. In 1966 he graduated with the degree of B.S. cum laude and since then has been in the Graduate School of the University of Florida. In December, 1972, he received the degree of Doctor of Philosophy with a major in physics.

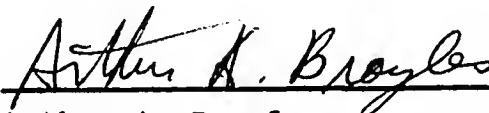
He is married to the former Miss Kwok-lan Chan, 陳國蘭, of Hong Kong. They have a seven-month-old boy, Yuan Tschang, 章元.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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Assistant Professor of Physics

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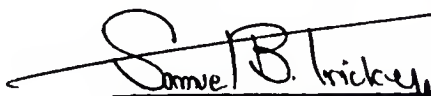
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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

A handwritten signature in dark ink, appearing to read "J. R. Sabin", is positioned above a horizontal line.

John R. Sabin
Associate Professor of Physics & Chemistry

This dissertation was submitted to the Department of Physics and Astronomy in the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1972


A. G. Smith

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